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On a Conjecture of Jackson on Non-homogeneous Quadratic Forms

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Here we prove the following modification of a conjecture of Jackson (*J. London Math. Soc.* (2) **3** (1971), 47–58) for indefinite quadratic forms of signature 0, ± 1 or ± 2. Let $Q(x_1,..., x_n)$ be a real indefinite quadratic form of determinant $D \neq 0$. Let $|\alpha| \leqslant |D|^{1/n}$. For any real numbers $a_1,..., a_n$, there exist $(x_1,..., x_n) \equiv (a_1,..., a_n)$ (mod 1) such that

$$|Q(x_1,..., x_n) - \alpha| \leqslant |D|^{1/n}.$$

In particular, the proof shows that we can find $(x_1,..., x_n) \equiv (a_1,..., a_n)$ (mod 1) such that

$$0 < Q(x_1,..., x_n) \leqslant 2\,|D|^{1/n}.$$

For forms of signature zero this result is also the best possible.

1. INTRODUCTION

Let $Q(x_1,..., x_n)$ be an indefinite quadratic form with real coefficients and determinant $D \neq 0$. Jackson [15] has proved that if $Q(x_1,..., x_n)$ is a zero form, then for any real numbers $a_1,..., a_n$ and α, there exist $(x_1,..., x_n) \equiv (a_1,..., a_n)$ (mod 1) such that

$$|Q(x_1,..., x_n) - \alpha| \leqslant |D|^{1/n}. \tag{1.1}$$

He conjectured that the result is probably true for all indefinite quadratic forms in $n \geqslant 4$ variables. Jackson also gave an example of an indefinite ternary quadratic form for which (1.1) does not hold for some α. It follows from the results of Blaney [4] that there are indefinite binary quadratic forms Q and real numbers α for which (1.1) is not solvable.

We observe that the results of Blaney [4] also show that (1.1 is solvable for all indefinite binary quadratic forms if $|\alpha| \leqslant |D|^{1/2}$ (see Lemma 4). The results of Dumir [10] show that for indefinite ternary forms, (1.1) is solvable

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if $|\alpha| \leqslant |D|^{1/3}$ (see Lemma 14). We are thus led to consider Jackson's conjecture with the restriction $|\alpha| \leqslant |D|^{1/n}$.

More precisely we consider

CONJECTURE A. *Let $Q(x_1,..., x_n)$ be a real indefinite quadratic form of determinant $D \neq 0$. Let $|\alpha| \leqslant |D|^{1/n}$. For any real numbers $a_1,..., a_n$ there exist $(x_1,..., x_n) \equiv (a_1,..., a_n)$ (mod 1) such that*

$$|Q(x_1,..., x_n) - \alpha| \leqslant |D|^{1/n}. \tag{1.2}$$

In this paper we shall prove

THEOREM 1. *Conjecture A is true for indefinite quadratic forms of signature 0, ± 1 or ± 2.*

Our proof will show that for non-zero forms of signature 0, ± 1 or ± 2, (1.2) holds with strict inequality. Combining with the result of Jackson [15], we have the following

THEOREM 2. *Let $Q(x_1,..., x_n)$ be a real indefinite quadratic form of determinant $D \neq 0$ and signature $\sigma = 0$, ± 1, or ± 2. Let $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta = 2|D|^{1/n}$. Then given any real numbers $a_1,..., a_n$ there exist $(x_1,...,x_n) \equiv (a_1,..., a_n)$ (mod 1) such that*

$$-\alpha < Q(x_1,..., x_n) \leqslant \beta. \tag{1.3}$$

For each such σ equality is needed in (1.3) for certain forms and suitable α and β.

Let $Q(x_1,...,x_n)$ be a real indefinite quadratic form of type (r, s), $0 < r < n$ and determinant $D \neq 0$. Blaney [3] has proved that there exist constants Γ independent of Q, such that, for any real numbers $a_1,..., a_n$, there exist $(x_1,..., x_n) \equiv (a_1,..., a_n)$ (mod 1) satisfying

$$0 < Q(x_1,..., x_n) \leqslant (\Gamma |D|)^{1/n}.$$

Let $\Gamma_{r,s}$ denote the greatest lower bound of all such constants Γ. Davenport and Heilbronn [17] showed that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ was proved independently by Blaney [5] and Barnes [1]. Dumir [8, 9] proved that $\Gamma_{1,2} = 8$, $\Gamma_{2,2} = 16$ and $\Gamma_{3,1} = 16/3$. Dumir and Hans-Gill [11] have shown that $\Gamma_{1,3} = 16$. Hans-Gill and Raka [12, 13] have proved that $\Gamma_{3,2} = 16$ and $\Gamma_{4,1} = 8$.

From Theorem 2 we can immediately deduce the following

THEOREM 3. (a) *For all positive integers* r, s *with* $r+s=n$ *and* $r-s=0, \pm 1$ *or* ± 2, *we have*

$$\Gamma_{r,s} \leqslant 2^n.$$

(b) $\Gamma_{r,r}=2^n$ *for all* $n=2r \geqslant 2$.

The first part is the special case $\alpha=0$ of Theorem 2. For the second part we observe that equality is necessary for the form $x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_{2r}^2$ and $(a_1,\ldots,a_{2r})=(\frac{1}{2},\ldots,\frac{1}{2})$.

We have also obtained precise values of $\Gamma_{r,s}$ for signatures ± 1, 2 and 3, which we hope to publish in due course.

2. An Auxiliary Result

If Q is an indefinite quadratic form in $n \geqslant 3$ variables which takes arbitrarily small non-zero values, then using the results of Oppenheim [17], Watson [20] has proved that we can solve (1.1) for all α, in fact with any $\varepsilon>0$ in place of $|D|^{1/n}$. So even the unrestricted Jackson conjecture is true for such forms. Thus for $n \geqslant 3$, in order to prove Theorem 1, it is enough to consider non-zero forms Q which do not take arbitrarily small values, i.e., for which the homogeneous minimum $m_H(Q)>0$, where

$$m_H(Q)=\operatorname{Inf}\{|Q(X)|: X \in Z^n, X \neq 0\}.$$

Replacing Q by $-Q$, if necessary, we cn suppose that the signature of Q is non-negative.

We shall deduce Theorem 1 from the following:

THEOREM 4. *Let* $n \geqslant \sigma+4$. *Suppose the following conditions hold*

I. *For any indefinite quadratic form* Q *in variables, with signature* σ *and determinant* $D \neq 0$, *for which* $m_H(Q)>0$, *we have*

$$Q \sim \psi(x_1+a_{12}x_2+\cdots, x_2+a_{23}x_3+\cdots)+Q'(x_3,\ldots,x_n), \tag{2.1}$$

where $\psi(x,y)$ *is an indefinite binary quadratic form of determinant* d *with* $|d|^{n/2}<|D|$.

II. *Conjecture A is true for indefinite quadratic forms in* $n-2$ *variables with signature* σ.

Then (1.2) *is true with strict inequality for forms* Q *in* n *variables, with signature* σ, *for which* $m_H(Q)>0$.

In this section we shall prove Theorem 4 and in the following sections we shall show that the hypotheses of Theorem 4 are satisfied for forms of signature $\sigma = 0$, 1 or 2, for all $n \geqslant \sigma + 4$, so that Theorem 1 follows.

We first recollect Lemma 4 of Birch [2].

LEMMA 1. *Let $\phi(x, y)$ be an indefinite binary form of determinant d. Then for any real numbers x_0, y_0 and μ we can find $(x, y) \equiv (x_0, y_0) \pmod 1$ such that*

$$|\phi(x, y) + \mu| \leqslant \max\left(\left(\frac{|d|}{2}\right)^{1/2}, |d|^{1/4} |\mu|^{1/2}\right).$$

Proof of Theorem 4. Let $Q(x_1, ..., x_n)$ be an indefinite real quadratic form of determinant D and signature σ and $m_H(Q) > 0$. Since (2.1) holds, we can suppose without loss of generality that

$$Q = \psi(x_1 + a_{12}x_2 + \cdots, x_2 + a_{23}x_3 + \cdots) + Q'(x_3, ..., x_n).$$

Then $Q'(x_3, ..., x_n)$ is an indefinite form in $(n-2)$ variables, with signature σ and determinant D/d. Since $|d|^n < |D|^2$, we have

$$|\alpha| \leqslant |D|^{1/n} < \left|\frac{D}{d}\right|^{1/(n-2)}.$$

Let $a_1, ..., a_n$ be any given real numbers. By hypothesis (II) there exist $(x_3, ..., x_n) \equiv (a_3, ..., a_n) \pmod 1$ such that

$$|Q'(x_3, ..., x_n) - \alpha| \leqslant \left|\frac{D}{d}\right|^{1/(n-2)}.$$

By Lemma 1 with $\phi(x, y) = \psi(x + a_{12}y, y)$, $\mu = Q'(x_3, ..., x_n) - \alpha$, $y_0 = a_2 + a_{23}x_3 + \cdots$, $x_0 + a_{12}y_0 = a_1 + a_{12}a_2 + a_{13}x_3 + \cdots$, we can find $(x, y) \equiv (x_0, y_0) \pmod 1$ satisfying

$$\begin{aligned} |\phi(x, y) + \mu| &\leqslant \max\left(\left(\frac{|d|}{2}\right)^{1/2}, |d|^{1/4} |\mu|^{1/2}\right) \\ &\leqslant \max\left(\left(\frac{|d|}{2}\right)^{1/2}, |d|^{1/4} \left|\frac{D}{d}\right|^{1/2(n-2)}\right). \end{aligned}$$

Thus there exist $(x_1, x_2) \equiv (a_1, a_2) \pmod 1$ such that

$$\begin{aligned} |Q(x_1, ..., x_n) - \alpha| &= |\phi(x, y) + \mu| \\ &\leqslant \max\left(\left(\frac{|d|}{2}\right)^{1/2}, |d|^{1/4} \left|\frac{D}{d}\right|^{1/2(n-2)}\right) < |D|^{1/n}, \end{aligned}$$

because $|d|^n < |D|^2$. This proves Theorem 4.

3. Forms of Signature Zero

Here we shall use the following:

Lemma 2. *Let $\phi(x,y)$ be an indefinite binary quadratic form of discriminant $\Delta^2(\Delta > 0)$. Let $\rho > 0$, $\sigma > 0$, $\rho\sigma > 1/16$. Then given any x_0, y_0 there exist $(x,y) \equiv (x_0, y_0)$ (mod 1) such that*

$$-\sigma\Delta < \phi(x,y) < \rho\Delta.$$

This is due to Davenport [6].

Lemma 3. *Let $\phi(x,y)$ be an indefinite quadratic form of discriminant $\Delta^2(\Delta > 0)$ and $0 \leqslant \mu \leqslant 1/3$. Then given any x_0, y_0 there exist $(x,y) \equiv (x_0, y_0)$ (mod 1) such that*

$$-\frac{\mu\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} < \phi(x,y) \leqslant \frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} .$$

Further strict inequality holds for $\mu = 0$ if ϕ is a non-zero form.

This is due to Blaney [4].

Lemma 4. *Conjecture A is true for indefinite binary forms. Further* (1.2) *holds with strict inequality for non-zero forms.*

Proof. Let $\phi(x,y)$ be an indefinite binary form of determinant D and discriminant $\Delta^2 = -4D(\Delta > 0)$. Let $|\alpha| \leqslant |D|^{1/2} = \Delta/2$. We have to show that given any x_0, y_0 there exist $(x,y) \equiv (x_0, y_0)$ (mod 1) such that

$$|\phi(x,y) - \alpha| \leqslant |D|^{1/2} = \Delta/2.$$

On replacing ϕ by $-\phi$, if necessary, we can suppose $\alpha \geqslant 0$. Let $\alpha = t\Delta$, so that $0 \leqslant t \leqslant 1/2$. Thus we have to solve

$$-(\tfrac{1}{2} - t)\Delta \leqslant \phi(x,y) \leqslant (\tfrac{1}{2} + t)\Delta \tag{3.1}$$

for $(x,y) \equiv (x_0, y_0)$ (mod 1).

If $0 \leqslant t \leqslant 1/4$, then $(\frac{1}{2} + t)(\frac{1}{2} - t) = \frac{1}{4} - t^2 > 1/16$, so that (3.1) is solvable by Lemma 2. If $1/4 < t \leqslant 1/2$, then taking $\mu = (\frac{1}{2} - t)(\frac{1}{2} + t)^{-1}$ in Lemma 3, we can find $(x,y) \equiv (x_0, y_0)$ (mod 1) such that

$$-\frac{(\frac{1}{2} - t)\Delta}{(5-8t)^{1/2}} < \phi(x,y) \leqslant \frac{(\frac{1}{2} + t)\Delta}{(5-8t)^{1/2}} .$$

Since $0 \leqslant t \leqslant \frac{1}{2}$, $(5-8t)^{1/2} \geqslant 1$ and (3.1) is satisfied. Further, strict inequality holds unless $t = 1/2$, i.e., $\mu = 0$. Also if $\mu = 0$ and ϕ is a non-zero form, strict inequality holds in Lemma 3 and hence here also. This proves Lemma 4.

LEMMA 5. *Hypothesis* I *of Theorem* 4 *is satisfied for quadratic forms Q in $n \geqslant 4$ variables of signature zero and determinant $D \neq 0$ with $m_H(Q) > 0$.*

This follows from Lemmas 9, 10 and 11 of Birch [2].

COROLLARY 1. *Conjecture A is true for forms of signature zero.*

In view of our ealier remarks this follows from Theorem 4 using induction on n and Lemmas 4 and 5.

4. HYPOTHESIS I FOR FORMS OF SIGNATURE 1 OR 2

In this section we shall prove that Hypothesis I of Theorem 4 is satisfied for forms of signature 1 or 2. We shall use the following lemmas:

LEMMA 6. *Let $f(X) = f(x_1, \ldots, x_n)$ be a quadratic form of determinant $D \neq 0$ and let $F(Y)$ be its adjoint form. Let $A = (u_1, \ldots, u_n) \in \mathscr{Z}^n$ with g.c.d. $(u_1, \ldots, u_n) = 1$ and $F(A) \neq 0$. Then*

$$f \sim f_0(x_1 + \cdots, \ldots, x_{n-1} + \cdots) + cx_n^2,$$

where $f_0(y_1, \ldots, y_{n-1})$ is a form of determinant $F(A)$.

Proof. Let $f(X) = X'MX$, then $F(Y) = (\det M)\, Y'M^{-1}Y$. Let T be a unimodular matrix such that $(0, \cdots 0, 1)\, T = A'$. Let $g(X) = f(T^{-1}X)$. Then $G(Y) = \operatorname{adj} g = F(T'Y)$, so that $F(A) = G(0, \ldots, 0, 1)$. Now

$$\begin{aligned} f \sim g &= g(x_1, \ldots, x_{n-1}, 0) + 2(a_{1n}x_1 + \cdots + a_{n-1,n}x_{n-1})\, x_n + a_{nn}x_n^2 \\ &= g_0(x_1, \ldots, x_{n-1}) + 2(a_{1n}x_1 + \cdots + a_{n-1,n}x_{n-1})\, x_n + a_{nn}x_n^2, \end{aligned}$$

$\det g_0(x_1, \ldots, x_{n-1}) = \det g(x_1, \ldots, x_{n-1}, 0) = G(0, \ldots, 0, 1) = F(A) \neq 0$. Now if

$$\begin{aligned} g_0(x_1, \ldots, x_{n-1}) &\sim \lambda_1(x_1 + \cdots)^2 + \lambda_2(x_2 + \cdots)^2 + \cdots + \lambda_{n-1}x_{n-1}^2 \\ &= f_0(x_1 + \cdots, \ldots, x_{n-1}), \end{aligned}$$

where $f_0(y_1,\dots,y_{n-1}) = \lambda_1 y_1^2 + \cdots + \lambda_{n-1} y_{n-1}^2$, then

$$\begin{aligned} g(x_1,\dots,x_n) &\sim \lambda_1(x_1 + \cdots)^2 + \cdots + \lambda_{n-1}x_{n-1}^2 \\ &\quad + 2(a_{1n}x_1 + \cdots + a_{n-1,n}x_{n-1})\,x_n + a_{nn}x_n^2 \\ &= \lambda_1(x_1 + \cdots + \beta_1 x_n)^2 + \lambda_2(x_2 + \cdots + \beta_2 x_n)^2 \\ &\quad + \cdots + \lambda_{n-1}(x_{n-1} + \beta_n x_n)^2 + cx_n^2 \\ &= f_0(x_1 + \cdots + \beta_1 x_n,\dots, x_{n-1} + \beta_n x_n) + cx_n^2. \end{aligned}$$

Clearly $\det f_0 = \det g_0 = f(A)$.

This proves the Lemma.

Remark 1. Let f be a form and F be its adjoint. If F is a multiple of a zero rational form so is f. To see this we notice that if F is a multiple of a rational form then so is adj F and since adj $F = (\det f)^{n-2} f$, f is a multiple of a rational form. Now suppose $f = X'MX$, where we can for simplicity suppose that M is a rational matrix. Then $F = (\det M)\,X'M^{-1}X$. If $F(A) = 0$, where $0 \neq A \in \mathscr{Z}^n$, then $f(M^{-1}A) = 0$ and f has the rational zero $M^{-1}A$.

LEMMA 7. *If $Q(x,y,z)$ is a ternary quadratic form of type* (2, 1) *and determinant $D \neq 0$, then there exist integers u, v, w with $(u, v, w) = 1$ such that*

$$0 < Q(u, v, w) \leqslant (\tfrac{4}{3}\,|D|)^{1/2},$$

except when $Q \sim \rho Q_i$, $1 \leqslant i \leqslant 8$, where Q_i are certain zero rational forms.

This follows from Theorem 2 of Watson [21].

LEMMA 8. *Let $f(x,y,z)$ be a non-zero ternary quadratic form of type* (2, 1) *and determinant $\Delta < 0$. Then*

$$f \sim \psi(x + \cdots, y + \cdots) + cz^2,$$

where $\psi(x,y)$ is an indefinite binary quadratic form of determinant d such that

$$|d|^{3/2} \leqslant \frac{2}{\sqrt{3}}\,|\Delta|.$$

Proof. Let $F = \operatorname{adj} f$, then $-F$ is a ternary form of type (2, 1) of determinant $-\Delta^2$. Since f is a non-zero form, it follows by Remark 1 that

$-F \not\sim \rho Q_i$, $1 \leqslant i \leqslant 8$. Therefore by Lemma 7 there exist integers u, v, w with $(u, v, w) = 1$ and

$$0 < -F(u, v, w) \leqslant \left(\frac{4}{3}|\Delta|^2\right)^{1/3} = \left(\frac{2}{\sqrt{3}}|\Delta|\right)^{2/3}.$$

Now the result follows from Lemma 6.

LEMMA 9. *Let $f(x, y, z, u)$ be a non-zero quaternary form of type* (3, 1) *and determinant $D < 0$. Then*

$$f \sim \psi(x + \cdots, y + \cdots) + a(z + \cdots)^2 + bu^2,$$

where $\psi(x, y)$ is an indefinite binary form of determinant d with

$$|d|^2 \leqslant \left(\frac{16}{9}\right)^{1/3} |D|. \tag{4.1}$$

Proof. Let $F = \operatorname{adj} f$. Since f is a non-zero form, F is not a multiple of a zero rational form. Thus either F is a zero incommensurable form or a non-zero form. Hence by Oppenheim [18] and Jackson [16] it follows that there exist integers x_0, y_0, z_0, u_0 with g.c.d. $(x_0, y_0, z_0, u_0) = 1$ satisfying

$$0 < -F(x_0, y_0, z_0, u_0) < |D|^{3/4}. \tag{4.2}$$

By Lemma 6

$$f \sim g(x + \cdots, y + \cdots, z + \cdots) + bu^2 = f_0(x, y, z, u),$$

where $\det g = F(x_0, y_0, z_0, u_0) < 0$, so that $g_0(x, y, z) = f_0(x, y, z, 0)$ is a (2, 1) form. Further, it is a non-zero form since f is so. Also $\det g = \det g_0$. By (4.2)

$$|\det g_0|^{4/3} = |F(x_0, y_0, z_0, u_0)|^{4/3} < |D|. \tag{4.3}$$

By Lemma 8

$$g_0(x, y, z) \sim \psi(x + \cdots, y + \cdots) + az^2,$$

where ψ is an indefinite binary form of determinant d with

$$|d|^{3/2} \leqslant \frac{2}{\sqrt{3}} |\det g_0|. \tag{4.4}$$

Thus

$$f \sim \psi(x + \cdots, y + \cdots) + a(z + \cdots)^2 + bu^2,$$

and from (4.3) and (4.4) we have

$$|d|^2 = |\det \psi|^2 \leqslant \left(\frac{2}{\sqrt{3}} |\det g_0|\right)^{4/3} < \left(\frac{16}{9}\right)^{1/3} |D|.$$

This proves Lemma 9.

Now to verify Hypothesis I of Theorem 4, we follow Birch [2], but give details for sake of clarity and completeness. We describe the process for the quadratic forms of any signature $\sigma \geqslant 0$ in $n \geqslant \sigma + 4$ variables.

Let $Q_n(x_1,..., x_n)$ be a non-zero form of signature $\sigma \geqslant 0$ and determinant $D \neq 0$ with $b = m_H(Q_n) > 0$. Let $0 < \varepsilon < \frac{1}{16}$ be sufficiently small. There exist integers $u_1,..., u_n$ such that g.c.d. $(u_1,..., u_n) = 1$ and $Q_n(u_1,..., u_n) = v$, such that

$$b \leqslant |v| < b(1 + \varepsilon). \tag{4.5}$$

By a unimodular transformation we can suppose that $Q_n(1, 0,..., 0) = v$, so that

$$Q_n \sim v(x_1 + a_{12}x_2 + \cdots)^2 - Q_{n-1}(x_2,..., x_n).$$

One can easily see that $m_H(Q_{n-1}) > 0$. Let

$$a = \operatorname{Inf}\{(\operatorname{sgn} v)\, Q_{n-1}(x_2,..., x_n): x_i \in \mathscr{Z}, (\operatorname{sgn} v)\, Q_{n-1}(x_2,..., x_n) > 0\}. \tag{4.6}$$

Then, as remarked above, $a > 0$. As before

$$Q_{n-1} \sim u(x_2 + \cdots)^2 - Q_{n-2},$$

where $a \leqslant (\operatorname{sgn} v)\, u = |u| < a(1 + \varepsilon)$. Thus we have

$$\begin{aligned} Q_n &\sim v(x_1 + a_{12}x_2 + \cdots)^2 - u(x_2 + \cdots)^2 + Q_{n-2}(x_3,..., x_n) \\ &= \psi(x_1 + \cdots, x_2 + \cdots) + Q_{n-2}(x_3,..., x_n), \end{aligned} \tag{4.7}$$

where $\psi(x, y) = vx^2 - uy^2$ is an indefinite binary form of determinant $\delta = -uv$ with $|\delta| < ab(1 + \varepsilon)^2$ and Q_{n-2} is a quadratic form of signature σ with $m_H(Q_{n-2}) > 0$. If $n - 2 \geqslant \sigma + 4$, we can repeat the above procedure.

LEMMA 10. *Let $Q_4(x, y, z, t)$ be a quadratic form of signature 0 with $m_H(Q_4) > 0$ and if we write*

$$Q_4(x, y, z, t) \sim \psi_1(x + \cdots, y + \cdots) + \psi_2(z, t),$$

where ψ_1 is obtained from Q_4 by the above process, then

$$|\det \psi_1| \leqslant \tfrac{25}{36} |\det \psi_2|. \tag{4.8}$$

This follows from Lemmas 7, 8 and 9 of Birch [2].

LEMMA 11. *Let $F(x_1,\dots,x_m)$ be an indefinite quadratic form of signature $\sigma \geqslant 0$ with $m_H(F) > 0$. Suppose*

$$F \sim \psi_1(x_1 + \cdots, x_2 + \cdots) + G(x_3,\dots,x_m),$$

and

$$G(x_3,\dots,x_m) \sim \psi_2(x_3 + \cdots, x_4 + \cdots) + H(x_5,\dots,x_m),$$

where $\psi_1 = v(x_1 + \cdots)^2 - u(x_2 + \cdots)^2$ is obtained by the procedure described above and ψ_2 is an indefinite binary form. Then

$$|\det \psi_1| \leqslant \tfrac{25}{36} |\det \psi_2|.$$

Proof. By hypothesis,

$$\begin{aligned} F &\sim \psi_1(x_1 + \cdots, x_2 + \cdots) + \psi_2(x_3 + \cdots, x_4 + \cdots) + H(x_5,\dots,x_m) \\ &= F_0(x_1,\dots,x_m), \end{aligned}$$

say. Consider

$$Q_4 = F_0(x_1,\dots,x_4,0,\dots,0) = \psi_1(x_1 + \cdots, x_2 + \cdots) + \psi_2(x_3 + \gamma x_4, x_4).$$

Q_4 is a quaternary form of signature 0 with $m_H(Q_4) > 0$. We now claim that while following the procedure discussed in the beginning of the section for Q_4, we can take $\psi_1(x_1 + \cdots, x_2 + \cdots)$ as its first binary section. Let $b' = m_H(Q_4)$. Then since $Q_4(1,0,\dots,0) = v$ and every value of Q_4 is a value of F_0 and hence of F, we have

$$b \leqslant b' \leqslant |v| < b(1+\varepsilon) \leqslant b'(1+\varepsilon),$$

where $b = m_H(F)$. Thus we can choose v to be the leading coefficient of Q_4. Thus

$$Q_4 = v(x_1 + \cdots + \alpha_{14}x_4)^2 - Q_3(x_2, x_3, x_4),$$

where

$$Q_3(x_2,x_3,x_4) = u(x_2 + \cdots)^2 - \psi_2(x_3,x_4).$$

Let

$$a' = \operatorname{Inf}\{(\operatorname{sgn} v)\, Q_3(x_2,x_3,x_4) \colon x_i \text{ integers, } (\operatorname{sgn} v)\, Q_3 > 0\}.$$

Every value of Q_3 is a value of $F_0 - v(x_1 + \cdots)^2$ and since $Q_3(1,0,0) = u$, we have

$$a \leqslant a' \leqslant (\operatorname{sgn} v)\, u = |u| < a(1+\varepsilon) \leqslant a'(1+\varepsilon),$$

so that we again choose u as the leading coefficient of Q_3, thus proving our claim. By Lemma 10, we have

$$|\det \psi_1| \leqslant \tfrac{25}{36} |\det \psi_2|.$$

This proves Lemma 11.

LEMMA 12. *Let $Q_n = Q(x_1,..., x_n)$ be an indefinite form in $n \geqslant 5$ variables of signature 1 and determinant $D \neq 0$, with $m_H(Q_n) > 0$. Then*

$$Q_n \sim \psi_1(x_1 + \cdots, x_2 + \cdots) + Q_{n-2}(x_3,..., x_n),$$

where $\psi_1(x, y)$ is an indefinite binary form of determinant d, such that

$$|d|^{n/2} \leqslant \frac{2}{\sqrt{3}} \left(\frac{5}{6}\right)^{(n-3)(n+1)/4} |D| < |D|.$$

Proof. Following the general procedure described above, we have

$$Q_n \sim \psi_1(x_1 + \cdots, x_2 + \cdots) + Q_{n-2}(x_3,..., x_n).$$

If $n - 2 \geqslant 5$, we repeat the above process and get

$$\begin{aligned} Q_n \sim\ & \psi_1(x_1 + \cdots, x_2 + \cdots) + \cdots + \psi_{(n-3)/2}(x_{n-4} + \cdots, x_{n-3} + \cdots) \\ & + Q_3(x_{n-2}, x_{n-1}, x_n), \end{aligned}$$

where $Q_3(x_{n-2}, x_{n-1}, x_n)$ is a ternary form of type (2, 1) which does not represent arbitrarily small values. Let its determinant be Δ. Then by Lemma 8

$$Q_3 \sim \psi_{(n-1)/2}(x_{n-2} + \cdots, x_{n-1} + \cdots) + cx_n^2,$$

where $\psi_{(n-1)/2}$ is an indefinite binary form of determinant $\delta_{(n-1)/2}$ satisfying

$$|\delta_{(n-1)/2}|^{3/2} \leqslant \frac{2}{\sqrt{3}} |\Delta|. \tag{4.9}$$

Let $\det \psi_i = \delta_i$. On applying Lemma 9 to $\psi_i + \psi_{i+1} + \cdots + cx_n^2$, we see that

$$|\delta_i| \leqslant \frac{25}{36} |\delta_{i+1}|, \qquad \text{for} \quad 1 \leqslant i \leqslant \frac{n-3}{2}. \tag{4.10}$$

Hence from (4.9) and (4.10) we have

$$\begin{aligned} |\delta_1|^{n/2} &= |\delta_1| \, |\delta_1|^{(n-2)/2} \\ &\leqslant \left(\frac{25}{36}\right)^{(n-2)/2} |\delta_1| \, |\delta_2|^{(n-2)/2} \end{aligned}$$

$$\leqslant \left(\frac{25}{36}\right)^{[(n-2)+(n-4)+\cdots+3]/2} |\delta_1| \cdots |\delta_{(n-3)/2}| \, |\delta_{(n-1)/2}|^{3/2}$$

$$\leqslant \frac{2}{\sqrt{3}} \left(\frac{5}{6}\right)^{(n-3)(n+1)/4} |D|$$

$$\leqslant \frac{2}{\sqrt{3}} \left(\frac{5}{6}\right)^{3} |D| \qquad (\text{since } n \geqslant 5)$$

$$< |D|.$$

This proves Lemma 12.

LEMMA 13. *Let $Q_n(x_1,\dots,x_n)$ be an indefinite quadratic form in $n \geqslant 6$ variables of signature 2 and determinant $D \neq 0$, with $m_H(Q_n) > 0$. Then*

$$Q_n \sim \psi(x_1 + \cdots, x_2 + \cdots) + Q_{n-2}(x_3,\dots,x_n),$$

where $\psi(x, y)$ is an indefinite binary form of determinant d with

$$|d|^{n/2} \leqslant \left(\frac{5}{6}\right)^{(n-4)(n+2)/4} \left(\frac{16}{9}\right)^{1/3} |D| < |D|.$$

Proof. Proceeding, as in Lemma 12, we have

$$Q_n \sim \psi_1 + \psi_2 + \cdots + \psi_{(n-4)/2} + Q_4(x_{n-3},\dots,x_n),$$

where ψ_i are binary forms and Q_4 is a quaternary form of type (3, 1) and determinant $\Delta \neq 0$ which does not take arbitrarily small values. By Lemma 9

$$Q_4 \sim \psi_{(n-2)/2}(x_{n-3} + \cdots, x_{n-2} + \cdots) + a(x_{n-1} + \cdots)^2 + bx_n^2,$$

where $\psi_{(n-2)/2}(x, y)$ is an indefinite binary form of determinant $\delta_{(n-2)/2}$ satisfying

$$|\delta_{(n-2)/2}|^2 \leqslant \left(\frac{16}{9}\right)^{1/3} |\Delta|. \tag{4.11}$$

Thus

$$Q_n \sim \psi_1 + \psi_2 + \cdots + \psi_{(n-2)/2} + a(x_{n-1} + \cdots)^2 + bx_n^2.$$

Then as in Lemma 12, we have

$$|\delta_i| \leqslant \frac{25}{36} |\delta_{i+1}|, \qquad \text{for} \quad 1 \leqslant i \leqslant \frac{n-4}{2}. \tag{4.12}$$

Hence using (4.11) and (4.12) we have

$$
\begin{aligned}
|\delta_1|^{n/2} &= |\delta_1|\,|\delta_1|^{(n-2)/2} \\
&\leqslant \left(\frac{25}{36}\right)^{(n-2)/2} |\delta_1|\,|\delta_2|^{(n-2)/2} \\
&\leqslant \left(\frac{25}{36}\right)^{(n-2)/2+\cdots+2} |\delta| \cdots |\delta_{(n-4)/2}|\,|\delta_{(n-2)/2}|^2 \\
&\leqslant \left(\frac{5}{6}\right)^{(n-4)(n+2)/4} \left(\frac{16}{9}\right)^{1/3} |\delta_1| \cdots |\delta_{(n-4)/2}|\,|\Delta| \\
&\leqslant \left(\frac{5}{6}\right)^{4} \left(\frac{16}{9}\right)^{1/3} |D| \qquad (\text{since } n \geqslant 6) \\
&< |D|.
\end{aligned}
$$

This proves Lemma 13.

Lemmas 12 and 13 show that Hypothesis I is satisfied for forms with signature 1 or 2.

5. Completion of the Proof of Theorem 1.

It is clear that Theorem 1 will now follow for forms of signature 1 or 2 by induction if we can prove that Hypothesis II of Theorem 4 is satisfied for forms of type (2, 1) and (3, 1). We prove it in this section. We first prove

Lemma 14. *Conjecture A is true for indefinite ternary forms f signature* 1. *Further strict inequality holds in* (1.2) *for non-zero forms.*

Proof. Let $Q(x, y, z)$ be an indefinite ternary form of signature 1 and determinant $D \neq 0$. It follows from Dumir [10] that, given any $t \geqslant 0$ and real numbers x_0, y_0, z_0, there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$, such that

$$-t(f(t)\,|D|)^{1/3} < Q(x, y, z) \leqslant (f(t)\,|D|)^{1/3}, \tag{5.1}$$

where $f(t) = 8/(1+t)^3$; equality is needed only for $t = 7$ and ∞ for some zero forms Q (one can easily verify that the functions stated in Dumir's theorem are $\leqslant 8/(1+t)^3$). (5.1) implies that

$$\left| Q(x, y, z) - \frac{1-t}{1+t}|D|^{1/3} \right| \leqslant |D|^{1/3}.$$

Further strict inequality holds if Q is a non-zero form. For every α with $|\alpha| \leqslant |D|^{1/3}$, we can choose $0 \leqslant t \leqslant \infty$ such that $\alpha = (1-t)(1+t)^{-1}|D|^{1/3}$ and Conjecture A follows for ternary forms.

Before taking up forms of type (3, 1) we observe that for zero forms conjecture A follows from Jackson [15]. Thus we need to prove it only for non-zero forms. We do so in

THEOREM 5. *Conjecture A is true for non-zero quaternary quadratic forms of signature* 2.

Proof. Let $Q(x, y, z, u)$ be a quaternary form of type (3, 1) and determinant $D \neq 0$. We have to show that if $|\alpha| \leqslant |D|^{1/4}$, then given any real numbers x_0, y_0, z_0, u_0 there exist $(x, y, z, u) \equiv (x_0, y_0, z_0, u_0) \pmod 1$, such that

$$|Q(x, y, z, u) - \alpha| < |D|^{1/4}. \tag{5.2}$$

By Theorem 1 of Jackson [16] there exist integers x_1, y_1, z_1, u_1 with g.c.d. $(x_1, y_1, z_1, u_1) = 1$, such that

$$0 < Q(x_1, y_1, z_1, u_1) = a < |D|^{1/4}.$$

On replacing Q by an equivalent form, we can suppose that $Q(1, 0, 0, 0) = a$, and write

$$Q = a\{(x + hy + gz + ku)^2 - \phi(y, z, u)\},$$

where $\phi(y, z, u)$ is ternary form of type (1, 2) and determinant $|D|\, a^{-4} > 1$.

Because of homogeneity it suffices to prove

THEOREM A. *Let $Q(x, y, z, u) = (x + hy + gz + ku)^2 - \phi(y, z, u)$, where $\phi(y, z, u)$ is a ternary form of type* (1, 2) *and determinant $D > 1$. Let $|\alpha| \leqslant |D|^{1/4}$. Then given any real numbers x_0, y_0, z_0, u_0 there exist $(x, y, z, u) \equiv (x_0, y_0, z_0, u_0) \pmod 1$ such that*

$$|Q(x, y, z, u) - \alpha) < |D|^{1/4}. \tag{5.3}$$

We shall use the following two lemmas in the proof of Theorem A.

LEMMA 15. *Let α, β, d be real numbers with $d > 1$. Let n be the integer given by $n < d \leqslant n + 1$. Then given any real number x_0 there exists $x \equiv x_0 \pmod 1$ such that*

$$0 < (x + \alpha)^2 + \beta < d,$$

provided

$$-n^2/4 < \beta < d - \tfrac{1}{4}.$$

This follows from Lemma 6 of Dumir [9].

LEMMA 16. *Let $\phi(y,z,u)$ be a ternary form of type* (1, 2) *and determinant $D>0$. Let $0<t<\frac{1}{9}$ and $g(t)>256/25(1-t)^3$. Then given any real numbers y_0, z_0, u_0 there exist $(y,z,u)\equiv(y_0,z_0,u_0)$* (mod 1), *such that*

$$t(g(t)\,|D|)^{1/3}<\phi(y,z,u)<(g(t)\,|D|)^{1/3}.$$

This follows from Theorem 2 of Dumir and Hans-Gill [11] and the Theorem of Jackson [15].

Proof of Theorem A. We have to show that if $|\alpha|\leqslant|D|^{1/4}$, there exist $(x,y,z,u)\equiv(x_0,y_0,z_0,u_0)$ (mod 1) such that $|Q(x,y,z,u)|-\alpha<|D|^{1/4}$, i.e.,

$$0<(x+hy+gz+ku)^2-\phi(y,z,u)-\alpha+|D|^{1/4}<2\,|D|^{1/4}. \tag{5.4}$$

Let $d=2\,|D|^{1/4}$. Since $D>1$, $d>2$, so that, if n is the integer such that $n<d\leqslant n+1$, then $n\geqslant 2$. By Lemma 15, (5.4) is satisfied if we can find $(y,z,u)\equiv(y_0,z_0,u_0)$ (mod 1) satisfying

$$-n^2/4<-\phi(y,z,u)-\alpha+|D|^{1/4}<d-1/4,$$

or

$$\frac{1}{4}-\frac{d}{2}-\alpha<\phi(y,z,u)<\frac{n^2}{4}+\frac{d}{2}-\alpha. \tag{5.5}$$

Note that $|\alpha|\leqslant|D|^{1/4}=d/2$, so that the right-hand side of (5.5) is positive.

Case i. $\frac{1}{4}-\frac{d}{2}-\alpha\leqslant 0$.

By Lemma 6, we see that (5.5) is satisfied if

$$\left(\frac{n^2}{4}+\frac{d}{2}-\alpha\right)-\left(\frac{1}{4}-\frac{d}{2}-\alpha\right)=\frac{n^2-1}{4}+d>2\,|D|^{1/3}=(d^4/2)^{1/3},$$

i.e.,

$$h(d)=(n^2-1+4d)\,d^{-4/3}>(32)^{1/3}. \tag{5.6}$$

Since $h(d)$ is a decreasing function of d and $d\leqslant n+1$, (5.6) is satisfied if

$$h(n+1)=(n+3)/(n+1)^{1/3}>(32)^{1/3}.$$

For $n\geqslant 2$, $(n+3)/(n+1)^{1/3}\geqslant 5.3^{-1/3}>(32)^{1/3}$. Hence (5.6) is satisfied and the result follows in Case *i*.

Case ii. $1/4-d/2-\alpha>0$.

Let

$$t = \frac{1/4 - d/2 - \alpha}{n^2/4 + d/2 - \alpha} = \frac{1 - 2d - 4\alpha}{n^2 + 2d - 4\alpha}.$$

Then $t > 0$. Since $|\alpha| \leqslant d/2$ and $d > n \geqslant 2$, we have

$$t = 1 - \frac{n^2 + 4d - 1}{n^2 + 2d - 4\alpha} \leqslant 1 - \frac{n^2 + 4d - 1}{n^2 + 4d} = \frac{1}{n^2 + 4d} < \frac{1}{9}.$$

Let $g(t) = (n^2/4 + d/2 - \alpha)^3 |D|^{-1}$. Then

$$(1 - t)^3 g(t) = \left(\frac{n^2 - 1}{4} + d\right)^3 |D|^{-1} = (n^2 - 1 + 4d)^3/4d^4.$$

The right-hand side is a decreasing function of d, and since $d \leqslant n + 1$ and $n \geqslant 2$ we have

$$(1 - t)^3 g(t) \geqslant (n + 3)^3/4(n + 1) \geqslant 125/12 > 256/25.$$

Hence by Lemma 16 there exist $(y, z, u) \equiv (y_0, z_0, u_0) \pmod 1$ such that

$$t(g(t)|D|)^{1/3} < \phi(y, z, u) < (g(t)|D|)^{1/3},$$

or

$$\frac{1}{4} - \frac{d}{2} - \alpha < \phi(y, z, u) < \frac{n^2}{4} + \frac{d}{2} - \alpha.$$

Thus (5.5) is satisfied in Case *ii* also and Theorem A follows.

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